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A FINITE DIFFERENT METHOD FOR PRICING OF EQUITY LOOKBACK OPTIONS.

ABSTRACT.

We try to clarify what "finite difference method" means in the financial literature. A finite difference method in mathematics is a numerical method for solving a partial differential equation (PDE). It seems that in the financial literature it is something different. We analyze this issue using as an example the pricing of Equity Lookback options.

Let's calculate the price of an Equity Price Lookback Call option (i.e. a European call option where the final spot price (that is compared to the strike price) is set to the maximum price that the underlying asset realizes over the life of the option) when the interest rate, dividend rate and implied volatility parameters of model (r, d, and σ respectively) are non-stochastic. In this case, the following PDE was suggested in [1]:

$$\frac{\partial P}{\partial t} = (r - d - 0.5\sigma^2)\frac{\partial P}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial s^2} - rP$$

(1)
where
P(s,L,t)=unknown function of 3 variables
s=Log(S)
S=spot price
t=time, measured from today to expiration
L=Maximum value of spot price S during the life of the option (i.e. from option trade date until expiration).

It is well known that the solution for equation (1) with any initial function $\phi(s,L)$ can be represented by the formula

$$P(s,L,t) = e^{-rt}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}\sigma}e^{-\frac{1}{2\sigma^2 t}(y-s-\mu t)^2}\phi(y,L)dy$$

$$\mu = r - d - 0.5\sigma^2$$

(2)
For an Equity Strike Lookback Call option, $\phi(s,L) = Max(L-K,0)$, where K=strike price. In this case, from (2) we have

$$P(s,L,t) = e^{-rt}Max(L-K,0)$$

(3)

Obviously, formulas (3) is not the complete solution of problem. Fortunately, for non-stochastic parameters we have analytical formulas for pricing Equity Price Lookback Call options:

$$P(S,L,t) = e^{-rt}(P1 + P2)$$

$$P1 = (K - L)(F(yL) - F(yK))$$

$$P2 = K(F(\infty) - F(yL)) - S * I$$

$$F(y) = e^{2\gamma y} Lp(-\frac{y + \gamma t}{\sqrt{t}}) - Lp(\frac{y - \gamma t}{\sqrt{t}})$$

$$I = \frac{4\gamma + 2\sigma}{\sigma + 2\gamma} e^{0.5t\sigma(\sigma + 2\gamma)} Lp(-\frac{yL - (\sigma + \gamma)t}{\sqrt{t}}) - \frac{2\gamma}{\sigma + 2\gamma} e^{(\sigma + 2\gamma)yL} Lp(-\frac{yL + \gamma t}{\sqrt{t}})$$

$$Lp(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2} du$$

$$\gamma = \frac{(r - d - 0.5\sigma^2)}{\sigma}$$

$$yL = \frac{1}{\sigma} Ln(\frac{L}{S})$$

$$yK = \frac{1}{\sigma} Ln(\frac{K}{S})$$

(4)

But for pricing this option, we can also use a recursive method based on an integral equation for the option price. This method is similar to a finite difference method, but not identical. At first inspection, the function P(t,s,L) must comply with the equation

$$\Pr(t, x, y) = \int_{-\infty}^{\infty} \Pr(t - \Delta t, z, Max / Min(z, y)) * prob(\Delta t, x, z)dz$$

(5)
where
x=logarithm of the spot price at time t
y= logarithm of maximum/minimum at time t
$prob(\Delta t, x, z)$ = the probability density of the transition of the logarithm of the equity price from x to z in time Δt.
Unfortunately for an Equity Price Lookback Call option, this equation is not correct, because the values x and z don't determine unambiguously the maximum/minimum in time of transition. Therefore, we must make the formula more complex. For given x and z, the maximum/minimum is a stochastic variable and the formula for the probability distribution of this variable can be written as

$$\Pr ob(x(0) = x, \max_{0 \le t \le T}(x(t)) = u, x(T) = z) = \frac{\partial}{\partial u}\left(-\frac{1}{\sigma\sqrt{2\pi t}} e^{2\frac{\mu}{\sigma^2}(u-x)} e^{-\frac{1}{2\sigma^2 t}(z+x-\gamma t-2u)^2}\right)$$

(6)

From (6)

$$\Pr ob(x(0) = x, \max_{0 \le t \le T}(x(t)) = u, x(T) = z) = P_d(x,u,z) = -2(\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2 t}(x + z - 2u)) * P$$

$$P = -\frac{1}{\sigma\sqrt{2\pi t}} e^{2\frac{\mu}{\sigma^2}(u-x)} e^{-\frac{1}{2\sigma^2 t}(z+x-\gamma t-2u)^2}$$

(7)

Using (7) we can write the correct formulas for the recursive calculation of the Equity Price Lookback Call option price:

$$\Pr(t,x,y) = (\Pr1 + \Pr2 + \Pr3 + \Pr4)e^{-rdt}$$

$$\Pr1 = \int_{-\infty}^{x} dz \int_{x}^{y} \Pr(t - \Delta t, z, y) * P_d(\Delta t, x, u, z)du$$

$$\Pr2 = \int_{-\infty}^{y} dz \int_{y}^{y} \Pr(t - \Delta t, z, u) * P_d(\Delta t, x, u, z)du$$

$$\Pr3 = \int_{x}^{y} dz \int_{z}^{y} \Pr(t - \Delta t, z, y) * P_d(\Delta t, x, u, z)du$$

$$\Pr4 = \int_{y}^{\infty} dz \int_{z}^{\infty} \Pr(t - \Delta t, z, u) * P_d(\Delta t, x, u, z)du$$

(8)

We assume that x and y are changing but within bounds and these bounds are calculated by formulas:

$$x_{min} = y_{min} = Ln(S(0)) + \mu(T_d)T_d - 5\sigma(T_d)\sqrt{T_d}$$

$$x_{max} = y_{max} = Ln(S(0)) + \mu(T_d)T_d + 5\sigma(T_d)\sqrt{T_d}$$

(9)

Formulas (8) are correct for any value of Δt.

Example: We used these formulas to calculate the price of an Equity Price Lookback Call option when the parameters of the model r, d and σ are non-stochastic and set to r= 0.115, d= 0.07 and σ =0.3. The other parameters are: S=100, K=100, L=100, t=0.5. The price calculated by analytical formulas (4) for this option is 18.37.
For calculation of the integrals in formulas (8) we used Gauss-Legendre formulas for numerical integration with 6 nodes. For variables x and y we used grids with n number of nodes. The initial value of the price equals:

$$\Pr(x(i), y(j),0) = Max(e^{y(j)} - K, 0)$$
$$i = 1,2......n$$
$$j = 1,2,.....n$$

For calculation of the price for Gauss-Legendre nodes we used double search function, i.e piece-wise constant interpolation. The results of the calculations are shown in Table 1.

Table 1

n	1000	500	200	100	50
Price	18.0089	17.9852	17.7297	17.4043	16.7423

These results were obtained using only for 1 time step, i.e. only for one node .The price calculation for many time steps required a large amount of computer time.
Conclusion:
1. The PDE of [1] for function P(s,L,t) is not accurate.
2. We can use the recursive method for pricing using an integral equation. But this is not a purely finite difference method, because we didn't use a PDE with 3 independent variables

Reference.

[1] D. Tavella and C. Randall (2000), Pricing Financial Instruments: The Finite Difference Method. Wiley
[2] Ioffe Mark (2003) - Binomial approach for look back options.